

**Operator Product Expansions and Consistency Relations in a  $O(N)$  Invariant  
Fermionic CFT for  $2 < d < 4$**

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**Abstract**

A conformally invariant theory of Majorana fermions in  $2 < d < 4$  with  $O(N)$  symmetry is studied using Operator Product Expansions and consistency relations based on the cancellation of shadow singularities. The critical coupling  $G_*$  of the theory is calculated to leading order in  $1/N$ . This value is then used to reproduce the  $O(1/N)$  correction for the anomalous dimension of the fermion field as evidence for the validity of our approach to conformal field theory in  $d > 2$ .

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## Introduction

Conformal field theory (CFT) methods have been proved powerful in describing the critical properties of systems in dimensions  $d > 2$  [1, 2, 3, 4]. Recently, [5, 6], we proposed a systematic treatment of the conformally invariant  $O(N)$  vector model in  $2 < d < 4$  based on Operator Product Expansions (OPE's) and consistency relations requiring the cancellation of shadow singularities. In [5, 6] we presented strong evidence that the proposed treatment gives the same results with the customary  $1/N$  expansion techniques for the anomalous dimensions of the fields in the  $O(N)$  vector model, (see [4] and references therein), while it proves more suitable for the evaluation of the important quantities  $C_T$  and  $C_J$ . It also unveils an interesting duality property of the model.

Our approach to the  $O(N)$  vector model in [5, 6] can be viewed as an effort to obtain dynamical information for a CFT in  $d > 2$  by algebraic methods, i.e. OPE's and consistency relations, without using a Lagrangian. If correct, our approach must be generally applicable to all CFT's in  $d > 2$ . In this letter we apply our algebraic approach to a fermionic  $O(N)$  invariant model in  $2 < d < 4$  and we show that it gives the same result with  $1/N$  expansion techniques for the anomalous dimension of the fermion field. The calculational details will be presented in a forthcoming publication [12].

## OPE's and Consistency Relations in a Fermionic $O(N)$ Invariant CFT in $2 < d < 4$

We consider a Euclidean fermionic CFT having as fundamental fields the  $N$ -component Majorana fermions  $\psi^\alpha(x)$ ,  $\bar{\psi}^\alpha(x)$ ,  $\alpha = 1, 2, \dots, N$ . The Majorana condition  $\psi(x) = C \bar{\psi}^T(x)$  ensures that the fermions are effectively real and the internal symmetry group at the fixed point maximal, the latter is taken here to be  $O(N)$ . We use Hermitean gamma matrices  $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu} \mathbf{1}$ . The charge conjugation matrix  $C$  is defined <sup>2</sup> through  $C^T = -C^{-1}$ ,  $C \gamma_\mu^T C^{-1} = -\gamma_\mu$ . The conformally invariant two-point function  $\langle \psi^\alpha(x_1) \bar{\psi}^\beta(x_2) \rangle$  takes the form [2, 3]

$$\langle \psi^\alpha(x_1) \bar{\psi}^\beta(x_2) \rangle = C_\psi \frac{\not{x}_{12}}{(x_{12}^2)^a} \delta^{\alpha\beta}, \quad \not{x}_{12} = \gamma \cdot x_1 - \gamma \cdot x_2, \quad a = \eta + \frac{1}{2}. \quad (1)$$

The normalisation constant  $C_\psi$  can be arbitrarily chosen and we henceforth set it to one <sup>3</sup>.

Discussions of CFT's usually concentrate on  $n$ -point functions of quasiprimary fields with  $n \geq 4$  since the functional form of the two- and three-point functions is fixed up to some arbitrary constants [5]. In this letter we consider the four-point function

$$\begin{aligned} \langle \psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2) \psi_k^\gamma(x_3) \bar{\psi}_l^\delta(x_4) \rangle &\equiv \Psi_{ijkl}^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) \\ &= \Psi_{ij,kl}(x_1, x_2; x_3, x_4) \delta^{\alpha\beta} \delta^{\gamma\delta} + \Psi_{in,ml}(x_1; x_3, x_2; x_4) C_{kn} C_{mj}^{-1} \delta^{\alpha\gamma} \delta^{\beta\delta} \\ &\quad - \Psi_{il,kj}(x_1, x_4; x_3, x_2) \delta^{\alpha\delta} \delta^{\gamma\beta}, \end{aligned} \quad (2)$$

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<sup>2</sup>See e.g. [7, 8].  $C$  has these properties for  $d = 2, 3, 4$  when the dimension of the gamma matrices is respectively 2, 2, 4. For general  $d$  we assume  $C^T = -C$  which is always possible if the representation of the gamma matrices is reducible.

<sup>3</sup>Generally speaking, in CFT the relative normalisation of two- and three-point functions depends on the dynamics. Our approach henceforth corresponds to fixing to unity the normalisation of the two-point functions and taking the normalisation of the three-point functions (couplings) to be the dynamical variables. Note however that the normalisation of the two-point functions of conserved currents, such as the energy momentum tensor  $T_{\mu\nu}(x)$ , is not arbitrary but it is fixed by the Ward identities of the theory [5, 6, 9].

where  $i, j, k, l, n, m$  are spinor indices. We may cast (2) in a more convenient for what follows form as

$$\begin{aligned}\Psi_{ijkl}^{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) &= F_{ijkl}^S(x_1, \dots, x_4) \delta^{\alpha\beta} \delta^{\gamma\delta} + F_{ijkl}^V(x_1, \dots, x_4) \frac{1}{2}(\delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma}) \\ &\quad + F_{ijkl}^T(x_1, \dots, x_4) \frac{1}{2}(\delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\alpha\delta} \delta^{\beta\gamma} - \frac{2}{N} \delta^{\alpha\beta} \delta^{\gamma\delta}),\end{aligned}\tag{3}$$

$$\begin{aligned}F_{ijkl}^S(x_1, \dots, x_4) &= \Psi_{ij,kl}(x_1; x_2, x_3; x_4) + \frac{1}{N} \left[ \Psi_{in,mj}(x_1; x_3, x_2; x_4) C_{kn} C_{mj}^{-1} \right. \\ &\quad \left. - \Psi_{il,kj}(x_1; x_4, x_3; x_2) \right],\end{aligned}\tag{4}$$

$$F_{ijkl}^V(x_1, \dots, x_4) = \Psi_{in,mj}(x_1; x_3, x_2; x_4) C_{kn} C_{mj}^{-1} + \Psi_{il,kj}(x_1; x_4, x_3; x_2),\tag{5}$$

$$F_{ijkl}^T(x_1, \dots, x_4) = \Psi_{in,mj}(x_1; x_3, x_2; x_4) C_{kn} C_{mj}^{-1} - \Psi_{il,kj}(x_1; x_4, x_3; x_2),\tag{6}$$

In correspondence with our treatment of the  $O(N)$  vector model, [5, 6], we suggest that we may evaluate (3) on inserting into it the OPE of  $\psi$  with  $\bar{\psi}$ . The (infinite) quasiprimary fields appearing in such an OPE are in principle unknown. However, as in [5, 6], we assume that this OPE is qualitatively similar to the free field theory one, at least as far as the most singular terms in the short distance limit are concerned. In practice this means that we can write down an ansatz for the leading terms in the OPE  $\psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2)$  based on a Taylor expansion for  $x_{12}^2 \rightarrow 0$  as in free field theory. It is crucial for our approach that a “low-lying scalar field”  $O(x)$ , which is  $O(N)$  singlet with dimension  $\eta_o$ ,  $0 < \eta_o < d$ , appears in the general OPE ansatz as it appears in free field theory. Substituting then our ansatz into (3) we obtain a short distance expansion for the four-point function. Such an expansion generally depends on a number of parameters, namely the couplings and the dimensions of the fields appearing in the OPE ansatz.

Next, we construct graphically the amplitude for the four-point function (3) in terms of skeleton graphs, having no self-energy or vertex insertions, with internal lines corresponding to the full two-point functions  $\langle \psi \bar{\psi} \rangle$  and another one, that of a scalar  $O(N)$  singlet field  $\tilde{O}(x)$  with dimension  $\tilde{\eta}_o$ ,  $0 < \tilde{\eta}_o < d$ . The latter field is related [5, 6] to the “low-lying scalar field” appearing in our OPE ansatz. Symmetry factors are determined as in the usual Feynman perturbation expansion. The triple vertices connecting the lines in such graphs <sup>4</sup> are the fully amputated three-point functions  $\langle \psi \bar{\psi} \tilde{O} \rangle$ , whose functional form being completely determined from conformal invariance depend only on the field dimensions and a coupling  $G_*$ . The important point of our graphical construction is that such graphs represent conformally invariant amplitudes as long as the dimensions of the fields involved in their construction satisfy a certain “uniqueness” condition [10]. Then, these graphs can in principle be calculated using the results in [11]. For the simplest cases we have found closed analytical expressions [5, 6, 12]. Consistency then of the graphical and the OPE evaluation of the four-point function (3) determines the couplings and the field dimensions in the theory in the context of a self-consistent  $1/N$  expansion.

Having outlined the general idea behind our approach to CFT in  $d > 2$  we concentrate on our fermionic theory. For general  $d$  an OPE ansatz consistent with  $O(N)$  and conformal symmetry can be taken as

$$\psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2) = \langle \psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2) \rangle$$

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<sup>4</sup>The skeleton graph expansion for the free theory corresponds simply to the disconnected graphs e.g. see [5].

$$\begin{aligned}
& + \sum_{O,J,S} \sum_n (\gamma_{[\mu_1, \dots, \mu_n]})_{ij} \left[ C_{[\mu_1, \dots, \mu_n]}^k(x_{12}, \partial_2) O_k(x_2) \delta^{\alpha\beta} \right. \\
& + \left. D_{[\mu_1, \dots, \mu_n]}^k(x_{12}, \partial_2) J_k^{\alpha\beta}(x_2) + E_{[\mu_1, \dots, \mu_n]}^k(x_{12}, \partial_2) S_k^{\alpha\beta}(x_2) \right], \quad (7)
\end{aligned}$$

where  $[\ ]$  denotes antisymmetrisation in the corresponding indices and  $k$  is a general spin label. The  $J_k^{\alpha\beta}(x)$  fields are antisymmetric and the  $S_k^{\alpha\beta}(x)$  are symmetric and traceless in the  $O(N)$  indices. Then, from (3) and (7) it follows that  $F_{ijkl}^S$ ,  $F_{ijkl}^V$  and  $F_{ijkl}^T$  receive separately contributions from the  $O(x)_k$ ,  $J_k^{\alpha\beta}(x)$  and  $S_k^{\alpha\beta}(x)_k$  fields.

Basically, we have expanded the left hand side of (7) in the complete basis  $\gamma_{[\mu_1, \dots, \mu_n]}$  of the antisymmetrised products of the gamma matrices i.e.  $\mathbf{1}_{ij}$ ,  $(\gamma_\mu)_{ij}$ ,  $(\gamma_{[\mu_1, \mu_2]})_{ij} = \frac{1}{2}[\gamma_{\mu_1}, \gamma_{\mu_2}]_{ij}$ , e.t.c. [8]. For general  $d$  this basis is infinite but it truncates when  $d = 2, 3, 4$ . The dimension of the gamma matrices is  $2^{d/2} \times 2^{d/2}$ ,  $(2^{(d-1)/2} \times 2^{(d-1)/2})$  for odd  $d$ , when  $d = 2, 3, 4$  but it is essentially arbitrary for non-integer  $d$  and in this case an infinite number of antisymmetrised products may be present.

A special role in the OPE (7) is played by the scalar  $O(N)$  singlet field  $O_0(x) \equiv O(x)$  with dimension  $\eta_o$ ,  $0 < \eta_o < d$ . Actually, a basic assumption concerning the form of the OPE (7) is to require the existence of *only one* such a field i.e. a “low-lying scalar field” giving more singular contributions than the energy momentum tensor <sup>5</sup> [5, 6]. The crucial point is then that from the conformally invariant form of the two- and three-point functions [3, 5, 6]

$$\langle O(x_1)O(x_2) \rangle = C_O \frac{1}{x_{12}^{2\eta_o}}, \quad (8)$$

$$\langle \psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2) O(x_3) \rangle = g_{\psi\bar{\psi}O} \frac{(\not{x}_{13} \not{x}_{23})_{ij}}{(x_{12}^2)^{\eta - \frac{1}{2}\eta_o} (x_{13}^2 x_{23}^2)^{\frac{1}{2}\eta_o + \frac{1}{2}}} \delta^{\alpha\beta} \equiv C_{ij}^0(x_{12}, \partial_2) \frac{C_O}{x_{23}^{2\eta_o}} \delta^{\alpha\beta}, \quad (9)$$

with  $g_{\psi\bar{\psi}O}$  the coupling, one can evaluate the OPE coefficients  $C_{ij}^0(x_{12}, \partial_2)$ , i.e. following techniques presented in detail in [5], as

$$\begin{aligned}
C_{ij}^0(x_{12}, \partial_2) &= \frac{g_{\psi\bar{\psi}O}/C_O}{B(\frac{1}{2}\eta_o + \frac{1}{2}, \frac{1}{2}\eta_o - \frac{1}{2})} \frac{1}{(x_{12}^2)^{a - \frac{1}{2}\eta_o - \frac{1}{2}}} \\
&\times (\mathbf{1} - \frac{1}{\eta_o - 1} \not{x}_{12} \not{\partial}_2)_{ij} \int_0^1 dt t^{\frac{1}{2}\eta_o - \frac{1}{2}} (1-t)^{\frac{1}{2}\eta_o - \frac{3}{2}} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{[-\frac{1}{4}t(1-t)x_{12}^2]^m}{(\eta_o + 1 - \mu)_m} \partial_2^{2m} e^{tx_{12}\partial_2}, \quad (10)
\end{aligned}$$

where  $(a)_m = \Gamma(\mu + a)/\Gamma(a)$ ,  $\mu = d/2$  and the derivatives on the r.h.s. of (10) are taken with constant  $|x_{12}|$ . Henceforth we set  $C_O = 1$ . Note that (10) is of order  $O(x_{12}^{\eta_o - 2\eta})$ , that is it includes the most singular contribution of  $O(x)$  in (7) as  $x_{12}^2 \rightarrow 0$ .

Substituting the OPE (7) into the four-point function (3) we can in principle find expressions for  $F_{ijkl}^S$ ,  $F_{ijkl}^V$  and  $F_{ijkl}^T$ . Using (10) we obtain for example

$$F_{ijkl}^S(x_1, \dots, x_4) = \frac{(\not{x}_{12})_{ij} \otimes (\not{x}_{34})_{kl}}{(x_{12}^2 x_{34}^2)^a} + g_{\psi\bar{\psi}O}^2 \frac{x_{24}^2}{(x_{12}^2 x_{34}^2)^a} \mathcal{A}_{ijkl}^{(\eta_o)}(x_1, \dots, x_4) + \dots, \quad (11)$$

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<sup>5</sup>Such a form for the OPE of CFT's in four dimensions has been recently discussed in [13].

$$\mathcal{A}_{ijkl}^{(\eta_o)}(x_1, \dots, x_4) = \left(1 - \frac{1}{\eta_o - 1} \not{x}_{12} \not{\partial}_2\right)_{ij} \otimes \left(1 - \frac{1}{\eta_o - 1} \not{x}_{34} \not{\partial}_4\right)_{kl} \left[ \mathcal{H}^{(\eta_o)}(u, v) \right], \quad (12)$$

$$\begin{aligned} \mathcal{H}_F^{(\eta_o)}(u, v) &= v^{\frac{1}{2}\eta_o + \frac{1}{2}} \sum_{n=0}^{\infty} \frac{v^n \left(\frac{1}{2}\eta_o + \frac{1}{2}\right)_n^2 \left(\frac{1}{2}\eta_o - \frac{1}{2}\right)_n^2}{n! (\eta_o + 1 - \mu)_n (\eta_o)_{2n}} \\ &\quad \times {}_2F_1\left(\frac{1}{2}\eta_o + \frac{1}{2} + n, \frac{1}{2}\eta_o + \frac{1}{2} + n; \eta_o + 2n; 1 - \frac{v}{u}\right), \end{aligned} \quad (13)$$

where we have used the usual invariant ratios  $u = (x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2)$ ,  $v = (x_{12}^2 x_{34}^2 / x_{14}^2 x_{23}^2)$ .

The second term on the r.h.s of (11) is the full contribution of the  $O(x)$  field to the four-point function. Here we shall only use the most singular term of this contribution in the limit as  $x_{12}^2, x_{34}^2 \rightarrow 0$ , or equivalently as  $u, v \rightarrow 0$ , namely

$$\mathcal{A}_{ijkl}^{(\eta_o)}(x_1, x_2, x_3, x_4) \rightsquigarrow \mathbf{1}_{ij} \otimes \mathbf{1}_{kl} v^{\frac{1}{2}\eta_o + \frac{1}{2}} + v^{\frac{1}{2}\eta_o + \frac{1}{2}} \mathcal{T}_{ijkl} O(|x_{12}|, |x_{34}|). \quad (14)$$

Another important field on the r.h.s of (7) is the  $O(N)$  conserved current appearing for example as<sup>6</sup>  $D^1(x_{12}, \partial_2) J_\mu^{\alpha\beta}(x_2)$  with dimension  $d - 1$  and spin 1. Its two-point function

$$\langle J_\mu^{\alpha\beta}(x_1) J_\nu^{\gamma\delta}(x_2) \rangle = C_J \frac{I_{\mu\nu}(x_{12})}{x_{12}^{\frac{2(d-1)}{2}}} \frac{1}{2} (\delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma}), \quad I_{\mu\nu}(x) = \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}, \quad (15)$$

is fixed from conformal invariance [5, 6] up to a proportionality constant  $C_J$  which has been proposed as a possible generalisation of the  $k$ -theorem in dimensions  $d > 2$  [6, 14]. From (7),  $J_\mu^{\alpha\beta}(x)$  contributes to the most singular term of  $F_{ijkl}^V$  in the short distance expansion as  $x_{12}^2, x_{34}^2 \rightarrow 0$ . To find this contribution we need to have an expression for the most singular term as  $x_{12}^2 \rightarrow 0$  of  $D^1(x_{12}, \partial_2) J_\mu^{\alpha\beta}(x_2)$ . Using the conformally invariant form of the three-point function<sup>7</sup>  $\langle \psi \bar{\psi} J_\mu \rangle$  we obtain

$$\psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2) \rightsquigarrow \frac{g_{\psi\bar{\psi}J}}{C_T} \frac{1}{(x_{12}^2)^{a-\mu-1}} (\gamma_\mu)_{ij} J_\mu^{\alpha\beta}(x_2) + \dots, \quad (16)$$

where the dots stand for less singular terms. From (16) we then obtain for the leading  $J_\mu^{\alpha\beta}(x)$  contribution

$$F_{ijkl}^V(x_1, x_2, x_3, x_4) \rightsquigarrow \frac{g_{\psi\bar{\psi}J}^2}{C_J} \frac{1}{(x_{12}^2 x_{34}^2)^{a-\mu}} (\gamma_\mu)_{ij} \otimes (\gamma_\nu)_{kl} \frac{I_{\mu\nu}(x_{24})}{x_{24}^{\frac{2(d-1)}{2}}} + \dots, \quad (17)$$

where again the dots stand for less singular terms. All that remains now for completing our calculation is to construct a conformally invariant graphical expansion for  $\Psi_{ijkl}^{\alpha\beta\gamma\delta}(x_1, \dots, x_4)$ , evaluate the amplitudes and compare the results with (11), (14) and (17).

<sup>6</sup>This term includes the most singular contribution of  $J_\mu^{\alpha\beta}(x)$  and it is all we need for our subsequent calculations.

<sup>7</sup>In general there are two independent conformal structures in the three-point function  $\langle \psi \bar{\psi} J_\mu \rangle$  [15, 16] and hence two independent coupling constants. The Ward identity gives a relation between these two constants. Then, our consistency requirement should presumably determine the remaining coupling constant *and* the important parameter  $C_J$  to leading order in  $1/N$ . This calculation will be presented in our more detailed forthcoming publication [12]. However, for the purposes of the present work we only need one of the two above mentioned terms, namely the one including the most singular contribution as  $x_{12}^2 \rightarrow 0$ , whose coupling constant is denoted by  $g_{\psi\bar{\psi}J}$ .

## The Free Theory

This is easily done for a theory of free massless  $N$ -component fermions where the four-point function (2) can be exactly calculated using Wick's theorem with elementary contraction (1). We easily obtain

$$F_{ijkl}^S(x_1, \dots, x_4) = \frac{(\not{x}_{12})_{ij} \otimes (\not{x}_{34})_{kl}}{(x_{12}^2 x_{34}^2)^a} - \frac{1}{N} \left[ \frac{(\not{x}_{14})_{il} \otimes (\not{x}_{32})_{kj}}{(x_{14}^2 x_{23}^2)^a} - \frac{(\not{x}_{13})_{in} \otimes (\not{x}_{24})_{mj}}{(x_{14}^2 x_{23}^2)^a} C_{kn} C_{mj}^{-1} \right] \quad (18)$$

$$F_{ijkl}^V(x_1, \dots, x_4) = \frac{(\not{x}_{14})_{il} \otimes (\not{x}_{32})_{kj}}{(x_{14}^2 x_{23}^2)^a} + \frac{(\not{x}_{13})_{in} \otimes (\not{x}_{24})_{mj}}{(x_{14}^2 x_{23}^2)^a} C_{kn} C_{mj}^{-1}. \quad (19)$$

To compare (18) and (19) with (11), (14) and (17) we use the Fierz identity and the properties of the charge conjugation matrix and we obtain for the most singular terms in the limit  $x_{12}^2, x_{34}^2 \rightarrow 0$

$$\begin{aligned} F_{ijkl}^S(x_1, \dots, x_4) &\rightsquigarrow \frac{(\not{x}_{12})_{ij} \otimes (\not{x}_{34})_{kl}}{(x_{12}^2 x_{34}^2)^a} + \frac{1}{(\text{Tr} \mathbf{1})N} \frac{[(x_{13} \cdot x_{24}) + (\frac{v}{u})^a (x_{14} \cdot x_{23})]}{(x_{13}^2 x_{24}^2)^a} \mathbf{1}_{ij} \otimes \mathbf{1}_{kl} + \dots, \\ &= \frac{(\not{x}_{12})_{ij} \otimes (\not{x}_{34})_{kl}}{(x_{12}^2 x_{34}^2)^a} + \frac{2}{(\text{Tr} \mathbf{1})N} \frac{x_{24}^2}{(x_{12}^2 x_{34}^2)^a} v^a \mathbf{1}_{ij} \otimes \mathbf{1}_{kl} + \dots, \end{aligned} \quad (20)$$

$$\begin{aligned} F_{ijkl}^V(x_1, \dots, x_4) &\rightsquigarrow \frac{2}{(\text{Tr} \mathbf{1})} \frac{x_{24}^2}{(x_{12}^2 x_{34}^2)^a} (uv)^{\frac{1}{2}a} \left( \frac{u}{v} \right)^{\frac{1}{2}a + \frac{1}{4}} (\gamma_\mu)_{ij} \otimes (\gamma_\nu)_{kl} I_{\mu\nu}(x_{24}) + \dots, \\ &= \frac{2}{(\text{Tr} \mathbf{1})} (\gamma_\mu)_{ij} \otimes (\gamma_\nu)_{kl} \frac{I_{\mu\nu}(x_{24})}{x_{24}^{4a-2}} + \dots, \end{aligned} \quad (21)$$

where we have used

$$\frac{x_{12}^2 x_{34}^2}{x_{24}^4} = (uv)^{\frac{1}{2}} + O(|x_{12}|, |x_{34}|), \quad \left( \frac{u}{v} \right)^k = 1 + O(|x_{12}|, |x_{34}|) \quad (22)$$

Consistency then of (21) with (17) requires

$$\eta = \mu - \frac{1}{2}, \quad \frac{g_{\psi\bar{\psi}J}^2}{C_J} = \frac{2}{(\text{Tr} \mathbf{1})}, \quad (23)$$

while from (18) and (12), (14) we deduce that

$$\eta_o = 2\eta = d - 1, \quad g_{\psi\bar{\psi}O}^2 = \frac{2}{(\text{Tr} \mathbf{1})N}, \quad (24)$$

in agreement with the usual free field theory results. Note that the field  $O(x)$  in the free field theory OPE (7) can be identified with the normal product  $:\bar{\psi}_i^\alpha(x)\psi_i^\alpha(x):/\sqrt{2N}$  as expected.

## The Non-Trivial Theory

As mentioned before, the amplitudes for the non-trivial theory are constructed in terms of skeleton graphs with internal lines corresponding to the full two-point functions  $\langle\psi\bar{\psi}\rangle$  and  $\langle\tilde{O}\tilde{O}\rangle$  while the triple vertices connecting the lines are fully amputated three-point functions. For example the triple vertex

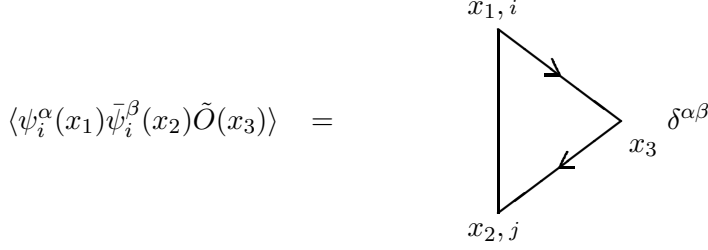


Figure 1: The graphical representation of  $\langle \psi_i^\alpha(x_1) \bar{\psi}_i^\beta(x_2) \tilde{O}(x_3) \rangle$ .

$V_{ij}^{\psi\bar{\psi}\tilde{O}}(x_1, x_2, x_3) \delta^{\alpha\beta}$  between,  $\psi_i^\alpha(x_1)$ ,  $\bar{\psi}_j^\beta(x_2)$  and  $\tilde{O}(x_3)$  is obtained on amputating all three legs from the full three-point function  $\langle \psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2) \tilde{O}(x_3) \rangle$ : the latter is analogous to (11) when  $\eta_o \rightarrow \tilde{\eta}_o$  and  $g_{\psi\bar{\psi}O} \rightarrow G_*$  with  $G_*$  the coupling. The three-point function may be graphically represented as shown in Fig.1. Then, using the inverse kernels

$$\left[ \langle \tilde{O}(x_1) \tilde{O}(x_2) \rangle \right]^{-1} = \rho(\tilde{\eta}_o) \frac{1}{(x_{12}^2)^{d-\tilde{\eta}_o}} \quad , \quad \rho(x) = \frac{1}{\pi^d} \frac{\Gamma(d-x)\Gamma(x)}{\Gamma(x-\mu)\Gamma(\mu-x)} . \quad (25)$$

$$\left[ \langle \psi_i^\alpha(x_1) \bar{\psi}_j^\beta(x_2) \rangle \right]^{-1} = p(a) \frac{(\not{x}_{12})_{ij}}{(x_{12}^2)^{d-a}} \delta^{\alpha\beta} \quad , \quad p(a) = \frac{a-1}{d-a+1} \rho(a-1) , \quad (26)$$

and the fermionic DEPP formula [11, 5, 6]

$$\int dx \frac{[\gamma \cdot (x-x_2)][\gamma \cdot (x-x_3)]}{(x-x_1)^{2a_1}(x-x_2)^{2a_2}(x-x_3)^{2a_3}} = \frac{\mathcal{F}(a_1; a_2, a_3)[\gamma \cdot (x_{12})][\gamma \cdot (x_{13})]}{(x_{23}^2)^{\mu-a_1}(x_{12}^2)^{\mu-a_3+1}(x_{13}^2)^{\mu-a_2+1}} , \quad (27)$$

$$\mathcal{F}(a_1; a_2, a_3) = \pi^\mu \frac{\Gamma(\mu-a_1)\Gamma(\mu-a_2+1)\Gamma(\mu-a_3+1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} , \quad (28)$$

which holds only for  $a_1 + a_2 + a_3 = d$ , we obtain

$$V_{ij}^{\psi\bar{\psi}\tilde{O}}(x_1, x_2, x_3) = \lambda_* \frac{(\not{x}_{13} \not{x}_{32})_{ij}}{(x_{12}^2)^{\mu-a+1}(x_{13}^2 x_{23}^2)^{\mu-\frac{1}{2}\tilde{\eta}_o+\frac{1}{2}}} , \quad (29)$$

$$\lambda_* = G_* \frac{1}{\pi^{3\mu}} \frac{\Gamma^3(\mu-\frac{1}{2}\tilde{\eta}_o+\frac{1}{2}) \Gamma^2(a) \Gamma(\tilde{\eta}_o) \Gamma(d-a-\frac{1}{2}+\frac{1}{2}\tilde{\eta}_o)}{\Gamma^3(\frac{1}{2}\tilde{\eta}_o+\frac{1}{2}) \Gamma^2(\mu-a+1) \Gamma(\mu-\tilde{\eta}_o) \Gamma(a-\mu-\frac{1}{2}+\frac{1}{2}\tilde{\eta}_o)} . \quad (30)$$

This amputation can be diagrammatically represented as shown in Fig.2 where the double directed line denotes the inverse kernel (26), the dotted line the inverse kernel (25), the lines ending in a circle are amputated and the dark blob denotes the full vertex (29). For simplicity we have factored out the  $O(N)$  indices from Fig.2

For the illustrative purposes of this letter and according to our proposed construction the first two terms in the graphical expansion of  $\Psi_{ij,kl}^{(\tilde{\eta}_o)}(x_1, x_2; x_3, x_4)$  in increasing order according to the number of vertices are as shown in Fig.3 where the superscript  $\tilde{\eta}_o$  denotes graphs built using as internal lines the full two-point function  $\langle \tilde{O}\tilde{O} \rangle$ . After some algebra whose details follow closely the calculations in [5, 6] we obtain

$$\mathcal{G}_{ij,kl}^{1,(\tilde{\eta}_o)}(x_1, x_2; x_3, x_4) = G_*^2 \frac{x_{24}^2}{(x_{12}^2 x_{34}^2)^a} \left[ \mathcal{A}_{ij,kl}^{(\tilde{\eta}_o)}(x_1, \dots, x_4) + C_F(d-\tilde{\eta}_o) \mathcal{A}_{ij,kl}^{(d-\tilde{\eta}_o)}(x_1, \dots, x_4) \right] , \quad (31)$$

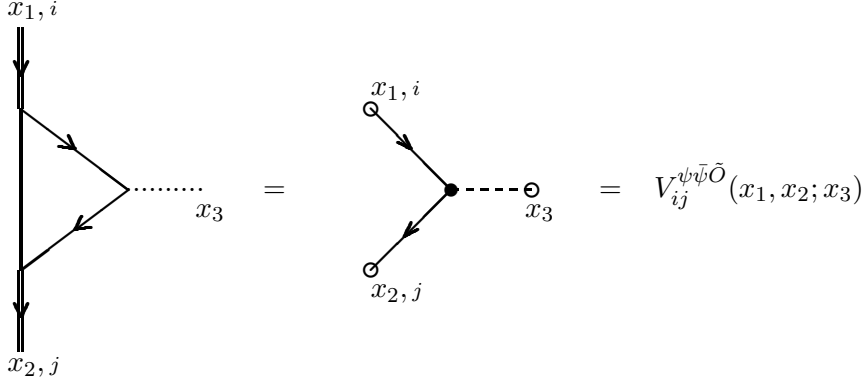


Figure 2: Amputating the three-point function

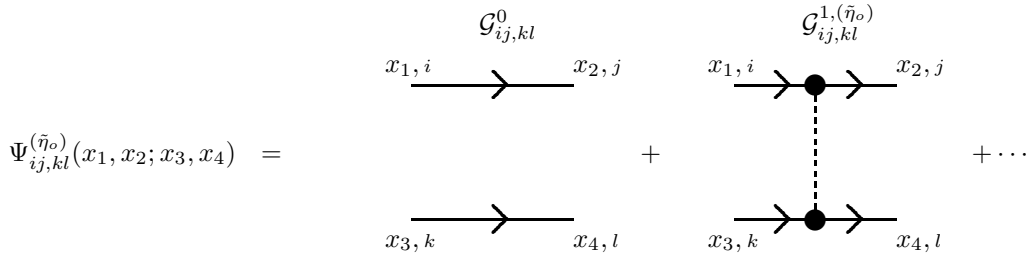


Figure 3: The Skeleton Graph Expansion for  $\Psi_{ij,kl}^{(\tilde{\eta}_o)}(x_1, x_2; x_3, x_4)$

with  $\mathcal{A}_{ij,kl}^{(\tilde{\eta}_o)}(u, v)$  as in (12) and

$$C_F(d - \tilde{\eta}_o) = C_F^{-1}(\tilde{\eta}_o) = \frac{\Gamma(\tilde{\eta}_o)\Gamma(\tilde{\eta}_o - \mu)\Gamma^4(\mu - \frac{1}{2}\tilde{\eta}_o + \frac{1}{2})}{\Gamma(d - \tilde{\eta}_o)\Gamma(\mu - \tilde{\eta}_o)\Gamma^4(\frac{1}{2}\tilde{\eta}_o + \frac{1}{2})}. \quad (32)$$

Note the similarity of (31) with the corresponding result for the one-particle exchange graphs in [5]. Basically, as pointed out in that reference, the amplitude for  $\mathcal{G}^{1,(\tilde{\eta}_o)}$  involves contributions from both  $\tilde{O}(x)$  and its *shadow field*<sup>8</sup>  $\tilde{O}_S(x)$  with dimension  $d - \tilde{\eta}_o$ .

From (4), (19) and (31) we obtain

$$\begin{aligned} F_{ijkl}^S(x_1, \dots, x_4) &= \frac{(\not{x}_{12})_{ij} \otimes (\not{x}_{34})_{kl}}{(x_{12}^2 x_{34}^2)^a} - \frac{1}{N} \left[ \frac{(\not{x}_{14})_{il} \otimes (\not{x}_{32})_{kj}}{(x_{14}^2 x_{23}^2)^a} - \frac{(\not{x}_{13})_{in} \otimes (\not{x}_{24})_{mj}}{(x_{14}^2 x_{23}^2)^a} C_{kn} C_{mj}^{-1} \right] \\ &+ G_*^2 \frac{x_{24}^2}{(x_{12}^2 x_{34}^2)^a} \left[ \mathcal{A}_{ij,kl}^{(\tilde{\eta}_o)}(x_1, \dots, x_4) + C_F(d - \tilde{\eta}_o) \mathcal{A}_{ij,kl}^{(d-\tilde{\eta}_o)}(x_1, \dots, x_4) \right] + \dots \end{aligned} \quad (33)$$

Equation (33) must agree with (11) and one immediately observes a remarkable correspondence: (11) and (33) contain all-order contributions in the expansion as  $x_{12}^2, x_{34}^2 \rightarrow 0$  with very similar closed analytic form. We refer to the second term on the r.h.s. of (11) and the first term in the second row

<sup>8</sup>For the notion of *shadow fields* which correspond to the *shadow symmetry* property of the conformal group in  $d > 2$  see [17] and references therein.



of (33). Therefore it seems natural to us to assume that these two terms are in fact equal, and this can be achieved if we identify  $g_{\psi\bar{\psi}O} \equiv G_*$  and  $\eta_o \equiv \tilde{\eta}_o$ . But then, from the results in the free field theory case, the bracketed term in the first row of (33) when expanded as  $x_{12}^2, x_{34}^2 \rightarrow 0$  must cancel the leading term of  $\mathcal{A}_{ij,kl}^{(d-\tilde{\eta}_o)}$  in the same limit. It is then easy to find that this consistency condition requires

$$G_*^2 \equiv g_{\psi\bar{\psi}O}^2 = -\frac{2}{N} C_F(\tilde{\eta}_o) \quad \text{and} \quad 2\eta = d - \tilde{\eta}_o \equiv d - \eta_o. \quad (34)$$

Equation (34) is the starting point for a self consistent solution of the theory. Firstly, since  $G_*^2 = O(1/N)$ , a consistent  $1/N$  expansion can be constructed: the order in  $1/N$  of the graphs simply corresponds to the number of triple vertices and at the same time we expand the couplings and the dimensions of the fields in a canonical part, (which equals their corresponding free field theory values), and  $1/N$  corrections. Secondly, the value for  $G_*^2$  in (34) serves as a “seed” in explicit calculations of the anomalous dimensions and other interesting quantities of the theory [5, 6]. Note that for  $0 < \tilde{\eta}_o < d$  and  $2 < d < 4$  then  $G_*^2 > 0$  which ensures the reality of the coupling in the graphically constructed theory at least to the order considered here.

Clearly, many interesting calculations can be done in the fermionic model in hand both to check our approach and also to obtain new interesting results [12]. Here we only present evidence that our approach is consistent with the standard  $1/N$  expansion results [2, 3] by evaluating the  $1/N$  correction to the anomalous dimension  $\eta$  of the fermion field. This is most easily done evaluating  $F_{ijkl}^V$  to  $O(1/N)$  using our graphical construction. This involves the evaluation of the “crossed” one-particle exchange graphs  $\mathcal{G}_{il,kj}^{1,(\tilde{\eta}_o)}(x_1; x_4, x_3; x_2)$  and  $\mathcal{G}_{in,ml}^{1,(\tilde{\eta}_o)}(x_1; x_3, x_2; x_2) C_{kn} C_{mj}^{-1}$  to leading order in  $1/N$ , i.e. using for the various parameters their free field theory values (23), (24). This calculation is done using techniques developed in [11, 5, 12] and the results are given in the Appendix. Then we set

$$\eta = \mu - \frac{1}{2} + \frac{1}{N} \eta_1, \quad (35)$$

and expand (21) in  $1/N$  when after the inclusion of the results in the Appendix we obtain <sup>9</sup>

$$\begin{aligned} F_{ij,kl}^V(x_1, \dots, x_4) &= \frac{2}{(\text{Tr} \mathbf{1})} \frac{x_{24}^2}{(x_{12}^2 x_{34}^2)^a} (uv)^{\frac{1}{2}\mu} \left[ 1 + \frac{1}{2N} \eta_1 \ln(uv) \right] (\gamma_\mu)_{ij} \otimes (\gamma_\nu)_{kl} I_{\mu\nu}(x_{24}), \\ &- \frac{G_*^2}{(\text{Tr} \mathbf{1})} \frac{(\mu-1)}{\Gamma(\mu-1)} a_{00} \frac{x_{24}^2}{(x_{12}^2 x_{34}^2)^a} (uv)^{\frac{1}{2}\mu} \ln(uv) (\gamma_\mu)_{ij} \otimes (\gamma_\nu)_{kl} I_{\mu\nu}(x_{24}) + \dots, \end{aligned} \quad (36)$$

with  $a_{nm}$  given in equation (46) of the Appendix and

$$G_*^2 = -\frac{2}{N} C_F(1) = \frac{2}{N} \frac{\Gamma(2\mu-1)}{\Gamma(2-\mu)\Gamma^3(\mu)}. \quad (37)$$

Consistency of the logarithmic  $O(1/N)$  terms in (36) and (17) then yields

$$\eta_1 = \frac{2}{(\text{Tr} \mathbf{1})} \frac{\Gamma(2\mu-1)}{\Gamma^2(\mu-1)\Gamma(2-\mu)\Gamma(\mu+1)}, \quad (38)$$

---

<sup>9</sup>Here we only present the logarithmic  $O(1/N)$  terms which are the relevant ones for the calculation of the anomalous dimension. The non-logarithmic  $O(1/N)$  terms are needed in the calculation of the couplings [5] and  $C_J$  [5, 12].

which is in agreement with the value of the critical exponent for the fermions in the Gross-Neveu and four-fermion models in  $2 < d < 4$  [2, 3, 18]. This shows that the our fermionic model is in the same universality class with the above two models in  $2 < d < 4$  as one expects.

## Summary and Outlook

We have outlined our approach to CFT in  $d > 2$  and applied it to a  $O(N)$  invariant fermionic model in  $2 < d < 4$ . Assuming the existence of a “low-lying scalar field” in the OPE  $\psi\bar{\psi}$  and constructing a graphical expansion for the four-point function  $\langle\psi\bar{\psi}\psi\bar{\psi}\rangle$  we obtained various consistency relations. These determine the leading order value in  $1/N$  of the critical coupling  $G_*$  in the theory. Using this value, we obtained the  $1/N$  correction to the anomalous dimension of the fermion field.

The calculations in the present work resemble in many ways the calculations in [5] where the  $O(N)$  invariant vector model was studied. Consequently, a unified picture for CFT’s in  $d > 2$  emerges, while agreement of (38) with the customary  $1/N$  expansion results is positive evidence for the validity of such a picture. Our approach emphasises the role played by the *shadow symmetry* of the conformal group in  $d > 2$ , for understanding CFT’s in more than two dimensions. Note that although our picture for CFT’s gives trivial results for  $d = 4$ , (e.g.  $\eta_1 = 0$  and  $G_*^2 = 2/(N \text{Tr}\mathbf{1})$  in  $d = 4$ ), a similar picture has been recently proposed for non-trivial four dimensional supersymmetric CFT’s [13]. A more detailed presentation of the calculations in the present letter and some new results concerning important quantities of the  $O(N)$  invariant fermionic model will be presented in a forthcoming publication [12].

## Acknowledgements

I am indebted to Hugh Osborn for his interest in my work and many useful suggestions.

## Appendix

We give here the analytic expressions for the “crossed” one-particle exchange graphs  $\mathcal{G}_{il,kj}^{1,(\tilde{\eta}_o)}$  and  $\mathcal{G}_{in,ml}^{1,(\tilde{\eta}_o)} C_{kn} C_{mj}^{-1}$  which are obtained using Symanzik’s method [11, 12] for conformal integration.

$$\begin{aligned} \mathcal{G}_{il,kj}^{1,(\tilde{\eta}_o)}(x_1; x_4, x_3; x_2) &= g_*^2 \frac{\Gamma(\tilde{\eta}_o)}{\Gamma(\mu - \tilde{\eta}_o)\Gamma^4(\frac{1}{2}\tilde{\eta}_o + \frac{1}{2})} \frac{1}{(x_{14}^2 x_{23}^2)^a}, \\ &\times \left[ \frac{x_{24}^2}{x_{14}^2 x_{23}^2} (\not{x}_{13} \not{x}_{43})_{il} \otimes (\not{x}_{13} \not{x}_{12})_{kl} \mathcal{I}_1(x_1; \frac{1}{2}\tilde{\eta}_o + 1, x_2; \mu - \frac{1}{2}\tilde{\eta}_o, x_3; \mu - \frac{1}{2}\tilde{\eta}_o + 1, x_4; \frac{1}{2}\tilde{\eta}_o), \right. \\ &+ \frac{x_{34}^2}{(x_{14}^2 x_{23}^2)^{\frac{1}{2}}} (\not{x}_{12} \not{x}_{42})_{il} \otimes (\not{x}_{13} \not{x}_{12})_{kj} \mathcal{I}_2(x_1; \frac{1}{2}\tilde{\eta}_o + 1, x_2; \mu - \frac{1}{2}\tilde{\eta}_o + 1, x_3; \mu - \frac{1}{2}\tilde{\eta}_o, x_4; \frac{1}{2}\tilde{\eta}_o), \\ &+ \frac{1}{x_{34}^2} (\not{x}_{13} \not{x}_{43})_{il} \otimes (\not{x}_{34} \not{x}_{24})_{kj} \mathcal{I}_2(x_1; \frac{1}{2}\tilde{\eta}_o, x_2; \mu - \frac{1}{2}\tilde{\eta}_o, x_3; \mu - \frac{1}{2}\tilde{\eta}_o + 1, x_4; \frac{1}{2}\tilde{\eta}_o + 1), \\ &+ \frac{1}{x_{24}^2} (\not{x}_{12} \not{x}_{42})_{il} \otimes (\not{x}_{34} \not{x}_{24})_{kj} \mathcal{I}_2(x_1; \frac{1}{2}\tilde{\eta}_o, x_2; \mu - \frac{1}{2}\tilde{\eta}_o + 1, x_3; \mu - \frac{1}{2}\tilde{\eta}_o, x_4; \frac{1}{2}\tilde{\eta}_o + 1), \\ &\left. - \left( \frac{1}{2} (\not{x}_{14} \gamma_\mu)_{il} \otimes (\gamma_\mu \not{x}_{32})_{kj} + \mathbf{1}_{il} \otimes (\not{x}_{13} \not{x}_{32})_{kj} + (\not{x}_{13} \not{x}_{43})_{il} \otimes \mathbf{1}_{kj} \right) \right] \end{aligned}$$

$$\times \mathcal{I}_0(x_1; \tfrac{1}{2}\tilde{\eta}_o, x_2; \mu - \tfrac{1}{2}\tilde{\eta}_o, x_3; \mu - \tfrac{1}{2}\tilde{\eta}_o, x_4; \tfrac{1}{2}\tilde{\eta}_o) \Bigg], \quad (39)$$

and

$$\begin{aligned} \mathcal{G}_{in,ml}^{1,(\tilde{\eta}_o)}(x_1; x_3, x_2; x_4) C_{kn} C_{mj}^{-1} &= g_*^2 \frac{\Gamma(\tilde{\eta}_o)}{\Gamma(\mu - \tilde{\eta}_o) \Gamma^4(\tfrac{1}{2}\tilde{\eta}_o + \tfrac{1}{2})} \frac{1}{(x_{13}^2 x_{24}^2)^a}, \\ &\times \left[ \frac{x_{24}^2}{x_{13}^2 x_{24}^2} (\not{x}_{14} \not{x}_{34})_{in} \otimes (\not{x}_{12} \not{x}_{14})_{ml} \mathcal{I}_2(x_1; \tfrac{1}{2}\tilde{\eta}_o + 1, x_2; \mu - \tfrac{1}{2}\tilde{\eta}_o, x_3; \tfrac{1}{2}\tilde{\eta}_o, x_4; \mu - \tfrac{1}{2}\tilde{\eta}_o + 1), \right. \\ &+ \frac{x_{34}^2}{(x_{13}^2 x_{24}^2)^{\frac{1}{2}}} (\not{x}_{12} \not{x}_{32})_{in} \otimes (\not{x}_{12} \not{x}_{14})_{ml} \mathcal{I}_2(x_2; \tfrac{1}{2}\tilde{\eta}_o + 1, x_2; \mu - \tfrac{1}{2}\tilde{\eta}_o + 1, x_3; \tfrac{1}{2}\tilde{\eta}_o, x_4; \mu - \tfrac{1}{2}\tilde{\eta}_o), \\ &+ \frac{1}{x_{23}^2} (\not{x}_{12} \not{x}_{32})_{in} \otimes (\not{x}_{32} \not{x}_{34})_{ml} \mathcal{I}_2(x_1; \tfrac{1}{2}\tilde{\eta}_o, x_2; \mu - \tfrac{1}{2}\tilde{\eta}_o + 1, x_3; \tfrac{1}{2}\tilde{\eta}_o + 1, x_4; \mu - \tfrac{1}{2}\tilde{\eta}_o), \\ &+ \frac{1}{x_{34}^2} (\not{x}_{14} \not{x}_{34})_{in} \otimes (\not{x}_{32} \not{x}_{34})_{ml} \mathcal{I}_2(x_1; \tfrac{1}{2}\tilde{\eta}_o, x_2; \mu - \tfrac{1}{2}\tilde{\eta}_o, x_3; \tfrac{1}{2}\tilde{\eta}_o + 1, x_4; \mu - \tfrac{1}{2}\tilde{\eta}_o + 1), \\ &- \left( \tfrac{1}{2} (\not{x}_{13} \gamma_\mu)_{in} \otimes (\gamma_\mu \not{x}_{24})_{ml} + \mathbf{1}_{in} \otimes (\not{x}_{32} \not{x}_{24})_{ml} + (\not{x}_{12} \not{x}_{32})_{in} \otimes \mathbf{1}_{mj} \right) \\ &\times \mathcal{I}_0(x_1; \tfrac{1}{2}\tilde{\eta}_o, x_2; \mu - \tfrac{1}{2}\tilde{\eta}_o, x_3; \tfrac{1}{2}\tilde{\eta}_o, x_4; \mu - \tfrac{1}{2}\tilde{\eta}_o) \Bigg] C_{kn} C_{mj}^{-1}, \quad (40) \end{aligned}$$

where the conformal integrals are

$$\mathcal{I}_0(x_1; a_1, x_2; a_2, x_3; a_3, x_4; a_4) \int_0^\infty d\lambda_1 \dots d\lambda_4 \prod_{i=1}^4 [\lambda_i^{a_i - \frac{1}{2}}] (S_\lambda)^{-\mu-1} \exp\left[-\frac{1}{S_\lambda} \sum_{i \neq j=1}^4 (\lambda_i \lambda_j x_{ij}^2)\right], \quad (41)$$

$$\mathcal{I}_2(x_1; b_1, x_2; b_2, x_3; b_3, x_4; b_4) = \int_0^\infty d\lambda_1 \dots d\lambda_4 \prod_{i=1}^4 [\lambda_i^{b_i - \frac{1}{2}}] (S_\lambda)^{-\mu-2} \exp\left[-\frac{1}{S_\lambda} \sum_{i \neq j=1}^4 (\lambda_i \lambda_j x_{ij}^2)\right], \quad (42)$$

$$S_\lambda = \sum_{i=1}^4 \lambda_i. \quad (43)$$

The integrals (41) and (42) are conformally invariant only if  $\sum_{i=1}^4 a_i = d$  and  $\sum_{i=1}^4 b_i = d + 2$  respectively. Here we only present the result for  $\mathcal{I}_0$  which is the only integral involved in the calculation of the leading terms in (39), (40) as  $x_{12}^2, x_{34}^2 \rightarrow 0$ .

$$\begin{aligned} \mathcal{I}_0(x_1; a_1, x_2; a_2, x_3; a_3, x_4; a_4) &= \frac{1}{(x_{23}^2)^{\mu-a_4+\frac{1}{2}} (x_{14}^2)^{a_1+\frac{1}{2}} (x_{34}^2)^{a_3+a_4-\mu} (x_{23}^2)^{a_2+a_3-\mu}} \\ &\times \left[ \sum_{n=0}^\infty \frac{v^n}{n!} \frac{\Gamma(a_3 + a_4 - \mu) \Gamma(\mu - a_3 + \tfrac{1}{2} + n) \Gamma(a_2 + \tfrac{1}{2} + n) \Gamma(\mu - a_4 + \tfrac{1}{2} + n) \Gamma(a_1 + \tfrac{1}{2} + n)}{(a_1 + a_2 + 1 - \mu)_n \Gamma(a_1 + a_2 + 1 + 2n)} \right. \\ &\times {}_2F_1(\mu - a_4 + \tfrac{1}{2} + n, a_1 + \tfrac{1}{2} + n; a_1 + a_2 + 1 + 2n; 1 - \frac{v}{u}) \\ &+ v^{a_3+a_4-\mu} \sum_{n=0}^\infty \frac{v^n}{n!} \frac{\Gamma(\mu - a_3 - a_4) \Gamma(a_4 + \tfrac{1}{2} + n) \Gamma(\mu - a_1 + \tfrac{1}{2} + n) \Gamma(a_3 + \tfrac{1}{2} + n)}{(a_3 + a_4 + 1 - \mu)_n \Gamma(a_3 + a_4 + 1 + 2n)} \end{aligned}$$

$$\times \Gamma(\mu - a_2 + \frac{1}{2} + n) {}_2F_1(a_3 + \frac{1}{2} + n, \mu - a_2 + \frac{1}{2} + n; a_3 + a_4 + 1 + 2n; 1 - \frac{v}{u}) \Big]. \quad (44)$$

For (39) and (40) we need (44) when  $a_1 + a_2 = a_3 + a_4 = \mu$  which yields

$$\begin{aligned} & \mathcal{I}_0(x_1; \frac{1}{2}\tilde{\eta}_o, x_2; \mu - \frac{1}{2}\tilde{\eta}_o, x_3; \mu - \frac{1}{2}\tilde{\eta}_o, x_4; \frac{1}{2}\tilde{\eta}_o) \\ &= \frac{1}{x_{23}^{\mu-\tilde{\eta}_o}} \frac{1}{(x_{14}^2 x_{23}^2)^{\frac{1}{2}\tilde{\eta}_o + \frac{1}{2}}} \sum_{n,m=0}^{\infty} \frac{v^n (1 - \frac{v}{u})^m}{n!m!} a_{nm} [-\ln v + b_{nm}], \end{aligned} \quad (45)$$

with

$$a_{nm} = \frac{\Gamma(\frac{1}{2}\tilde{\eta}_o + \frac{1}{2} + n) \Gamma(\mu - \frac{1}{2}\tilde{\eta}_o + \frac{1}{2} + n) \Gamma(\mu - \frac{1}{2}\tilde{\eta}_o + n + m) \Gamma(\frac{1}{2}\tilde{\eta}_o + \frac{1}{2} + n + m)}{\Gamma(1 + n) \Gamma(\mu + 1 + 2n + m)}, \quad (46)$$

$$\begin{aligned} b_{nm} &= 2\Psi(1 + n) + 2\Psi(\mu + 1 + 2n + m) - \Psi(\frac{1}{2}\tilde{\eta}_o + \frac{1}{2} + n) - \Psi(\mu - \frac{1}{2}\tilde{\eta}_o + \frac{1}{2} + n) \\ &\quad - \Psi(\mu - \frac{1}{2}\tilde{\eta}_o + \frac{1}{2} + n + m) - \Psi(\frac{1}{2}\tilde{\eta}_o + \frac{1}{2} + n + m), \end{aligned} \quad (47)$$

where  $\Psi(x) = \Gamma'(x)/\Gamma(x)$ .  $\mathcal{I}_0(x_1; \frac{1}{2}\tilde{\eta}_o, x_2; \mu - \frac{1}{2}\tilde{\eta}_o, x_3; \frac{1}{2}\tilde{\eta}_o, x_4; \mu - \frac{1}{2}\tilde{\eta}_o)$  is obtained from (45) setting  $x_3 \leftrightarrow x_4$ .

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